Exercise sheet 5: Stress-energy conservation, gravitational Doppler, and parallel transport

Please prepare your solutions, ready to present in the class on **25.05.2022** at **16:00**.

1. The stress-energy tensor for an electromagnetic field with field strength F_{ab} is
$$T_{ab} = F_{ac}F^c_b - \tfrac{1}{4}g_{ab}F_{cd}F^{cd}.$$
   (a) Show that the stress-energy tensor is traceless.
   (b) Show that the vacuum electromagnetic field equations $\nabla^a F_{ab} = 0$ and $\nabla_{[a}F_{bc]} = 0$ imply covariant conservation of the stress-energy tensor.

2. A metric of the form $g_{\mu\nu} = \mathrm{e}^{2\lambda}\eta_{\mu\nu}$, with $\lambda = \lambda(x^\mu)$ some real-valued function on spacetime, and $\eta_{\mu\nu}$ the usual Minkowski metric, is called *conformally flat*.
   (a) Compute the Christoffel symbols $\Gamma^\gamma_{\mu\nu}$ for a conformally flat metric.
   (b) What paths do light rays follow in a conformally flat spacetime?

3. The four-dimensional anti-de Sitter (AdS) metric may be described by the line-element
$$\mathrm{d}s^2 = \frac{u^2}{R^2}\left(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right) + \frac{R^2}{u^2}\,\mathrm{d}u^2.$$
   Here $(t, x, y, u) \in \mathbb{R}^3 \times [0, \infty)$ are *Poincaré* coordinates, and $R > 0$ is a constant.
   (a) Determine the AdS metric in new coordinates (t, x, y, z) where $z = \frac{R^2}{u}$.
   (b) A static observer at $x = y = 0$, and $z = R$ shines a light with energy E_1 towards a second static observer at $x = y = 0$, and $z = 2R$. What is the energy E_2 of the light as measured by the second observer? Does the light redshift or blueshift?

4. Consider the conical surface K defined by coordinates (r, ϕ) and with metric components
$$g_{rr} = 1 + 4r^2, \quad g_{\phi\phi} = r^2, \quad g_{r\phi} = g_{\phi r} = 0.$$
   Let $c(s) = (1, s)$, $s \in [0, 2\pi]$ be a curve on K.
   (a) Is c a geodesic?
   (b) Determine formulae for $a_1(s), a_2(s)$ such that $v = a_1\partial_r + a_2\partial_\phi$ is parallel along c.
   (c) Determine, in terms of the basis $(\partial_r, \partial_\phi)$, the linear map $P_c : T_{(1,0)}K \longrightarrow T_{(1,2\pi)}K$ describing parallel transport of vectors along c.